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THE STRESS CONCENTRATION AROUND A SEMI-INFINITE CYLINDRICAL CRACK DURING THE SHOCK LOADING OF AN ELASTIC MEDIUM BY A CENTRE OF ROTATION†

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The problem of the concentration of elastic stresses in an unbounded elastic medium containing a semi-infinite cylindrical crack is solved using the method of discontinuous solutions. The method enables the problem to be reduced to a one-dimensional integrodifferential equation in the unknown jump in displacements in Laplace transform space. The solution of the equation obtained is based on the combined use of orthogonal functions and time sampling of the equation, which leads to an infinite system of linear algebraic equations. As a result, a wave field disturbed by a crack is constructed, and a formula is obtained for calculating the stress intensity factor. The wave field is analysed numerically, and a graph of the stress intensity factor as a function of time is plotted. © 2001 Elsevier Science Ltd. All rights reserved.

The range of application of the spectral relationship obtained earlier [1] for Chebyshev–Laguerre polynomials to non-stationary problems of fracture mechanics is widening. An example of the use of this relationship to construct an accurate solution for one such problem has been given, and the effectiveness of using it to construct an approximate asymptotic solution for more complex problems of fracture mechanics has been painted out [1]. However, the method proposed in [1] is only suitable for those fracture mechanics problems in which the solving integrodifferential equation, written in Laplace transforms, contains a prescribed right-hand part that can be expanded in inverse powers of the Laplace transformation parameter. However, in many problems, the latter does not occur. A problem of this type is the determination of the stress intensity factor at the edge of a semi-infinite crack in an unbounded elastic medium loaded at the origin of coordinates with a shock centre of rotation.

The solution of this problem required considerable modification of the method used in [1], which is set out below. A different method was proposed for solving non-stationary problems of mechanics in [2, 3]. Using a method [4] proposed for the related static problem, it would be possible to obtain, for Laplace transforms of the functions sought, an accurate solution in the form of quadratures containing infinite derivatives. However, the numerical realization of such a solution, even at this stage, is far from a simple problem; it becomes even more complicated if an attempt is made to change from the Laplace transform to the originals.

1. FORMULATION OF THE PROBLEM AND ITS REDUCTION TO A ONE-DIMENSIONAL INTEGRODIFFERENTIAL EQUATION

An unbounded elastic medium ($0 < r < \infty, -\pi < \varphi < \pi, -\infty < z < \infty$) with shear modulus G and Poisson's ratio μ, containing a semi-infinite crack, the surface of which is described by the relations

$$r = R, \quad -\pi < \varphi < \pi, \quad a < z < \infty \tag{1.1}$$

is subject to the shock action of a centre of rotation at the origin of coordinates with a moment $M(t) = MH(t)$, where $H(t)$ is the Heaviside unit function. The sides of the crack $r = R - 0$ and $r = R + 0$ are considered to be stress-free

$$\tau_{r\varphi}(R \pm 0, z, t) \equiv \tau_{\varphi}(R \pm 0, z, t) = 0 \tag{1.2}$$

It is required to determine the stress state and wave fields under zero initial conditions.

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The displacement $u_\varphi(r, z, t) \equiv u(r, z, t)$ satisfies the equation [5]

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad c^2=\frac{G}{\rho} \tag{1.3}$$

and the stress $\tau_{r\varphi} = \tau_\varphi$ is expressed by the formula

$$\tau_\varphi(r,z,t)=G[u'(r,z,t)-r^{-1}u(r,z,t)] \tag{1.4}$$

Here and below, a prime denotes a derivative with respect to the variable r.

The displacement and stress fields are written in the form

$$u=u^0+u^1, \quad \tau_\varphi=\tau_\varphi^0+\tau_\varphi^1 \tag{1.5}$$

where u^0 and τ^0 are the displacements and stresses caused by the centre of rotation when there is no crack in the elastic medium, and u^1 and τ^1 are the required disturbed displacement and stress fields caused by the presence of the crack (1.1). Taking (1.5) into account, we will write the condition at the crack (1.2) in the form

$$\tau_\varphi^1(R\mp 0,z,t)=-\tau_\varphi^0(R,z,t) \tag{1.6}$$

The disturbed field u^1 is constructed in the form of the discontinuous solution of Eq. (1.3) for defect (1.1) [6]. To construct it, integral Laplace and Fourier transformations are used successively in accordance with the classical scheme

$$u_p^1(r,z)=\int_0^\infty e^{-pt}u^1(r,z,t)dt, \quad u_{p\lambda}^1(r)=\int_{-\infty}^\infty e^{i\lambda z}u_p^1(r,z)dz$$

and then an integral Hankel transformation with respect to r

$$u_{p\lambda\alpha}^1=\int_0^\infty rJ_1(\alpha r)u_{p\lambda}^1(r)dr$$

in accordance with the generalized scheme [6].

After inverting the Hankel and Fourier transformations, the Laplace transform of the required discontinuous solution is obtained in the form

$$\begin{aligned} u_p^1(r,z)=&\int_a^\infty \left\langle u_p^{1\prime}(R,\zeta)\right\rangle \int_{-\infty}^\infty e^{-i\lambda(z-\zeta)}G_{p\lambda}(r,R)d\lambda d\zeta- \\ &-\int_a^\infty \left\langle u_p^1(R,\zeta)\right\rangle \int_{-\infty}^\infty e^{-i\lambda(z-\zeta)}\frac{\partial}{\partial R}G_{p\lambda}(r,R)d\lambda d\zeta \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} &\left\langle u_p^1(R,\zeta)\right\rangle=u_p^1(R-0,\zeta)-u_p^1(R+0,\zeta) \\ &\left\langle u_p^{1\prime}(R,\zeta)\right\rangle=u_p^{1\prime}(R-0,\zeta)-u_p^{1\prime}(R+0,\zeta) \\ &G_{p\lambda}(r,R)=\begin{cases} I_1(r\sqrt{\lambda^2+p^2})K_1(R\sqrt{\lambda^2+p^2}), & r<R \\ I_1(R\sqrt{\lambda^2+p^2})K_1(r\sqrt{\lambda^2+p^2}), & r>R \end{cases} \end{aligned} \tag{1.8}$$

The jump in the derivative of the required function in (1.7) can be eliminated by using (1.4) and (1.6). As a result we will have

$$\left\langle u_p^{1\prime}(R,z)\right\rangle=\left\langle u_p^1(R,z)\right\rangle \tag{1.9}$$

The use of this equality enables us not only to eliminate the jump in the derivative of the required function but also to confine ourselves to satisfying condition (1.6) on only one of the sides of the crack, for example with $R - 0$. As a result, we arrive at the integrodifferential equation

$$\left(-\frac{d^2}{dz^2}+p^2\right)\int_a^\infty X_p(\zeta)k_p(z-\zeta)d\zeta=-\tau^0_{p\varphi}(R,z),\quad 0\leq z<\infty \tag{1.10}$$

Here

$$X_p(\zeta)=\left\langle u_p^1(R,\zeta)\right\rangle$$

$$k_p(z-\zeta)=\int_a^\infty \cos\lambda(z-\zeta)I_2(\sqrt{\lambda^2+p^2})K_2(\sqrt{\lambda^2+p^2})d\lambda \tag{1.11}$$

$$\tau^0_{p\varphi}=\frac{Mz}{(r^2+z^2)^{5/2}}\exp(-pc(\sqrt{r^2+z^2}-R))\frac{(\sqrt{r^2+z^2}\,p^2c^2-pc+3)}{p(p-p_1)(p-p_2)}$$

$$u^0_p=\frac{Mz}{2(r^2+z^2)^{3/2}}\exp(-pc(\sqrt{r^2+z^2}-R))\frac{(\sqrt{r^2+z^2}\,pc-3)}{p(p-p_1)(p-p_2)}$$

where $p_1=(\sqrt{3}/2)(i-\sqrt{3})c, p_2=-(\sqrt{3}/2)(i+\sqrt{3})c$. Expressions for $\tau^0_{p\varphi}$ and u^0_p were obtained earlier [7].

2. A METHOD OF SOLVING THE INTEGRODIFFERENTIAL EQUATION

We introduce the variables

$$z=a+xR,\quad \xi=a+\eta R$$

into (1.10).

The irregular part of the kernel is separated out as follows. The range of integration is divided into two sections: $(0,\infty)=(0,A)+(A,\infty)$, where A is a fairly large positive number, and in the second section, instead of the derivative of cylindrical functions, the principal term of their asymptotic representation for large values of the argument is selected. The subsequent use of formula 7.12(27) from [4] of the integral representation of the MacDonald function $K_0(x)$ enables the required representation to be obtained.

$$k_p(x-\eta)=K_0(pR(x-\eta))+R_p(R(x-\eta)) \tag{2.1}$$

where the regular part of the nucleus has the form

$$R_p(R(x-\eta))=$$
$$=\int_0^A \cos\lambda R(z-\eta)\left[I_2(\sqrt{\lambda^2+p^2})K_2(\sqrt{\lambda^2+p^2})-\frac{1}{\sqrt{\lambda^2+p^2}}\right]d\lambda \tag{2.2}$$

Then, Eq. (1.10) takes the form

$$\left(-\frac{d^2}{dx^2}+p^2\right)\int_0^\infty X_p(a+\eta R)\big(K_0(pR(x-\eta))+R_p(R(x-\eta))\big)d\eta=-\tau^0_{p\varphi}(R,a+xR) \tag{2.3}$$

Temporarily assuming the parameter p to be positive, we make the replacement

$$pxR=\xi,\quad pR\eta=\sigma \tag{2.4}$$

in (2.3).

As a result, we arrive at the equation

$$\left(1-\frac{d^2}{d\xi^2}\right)\int_0^\infty pX_p(a+\sigma p^{-1})\left(K_0(|\xi-\sigma|)+R_p\left(|\xi-\sigma|p^{-1}\right)\right)d\sigma =$$
$$= -\tau^0_{p\varphi}(R, a+\xi p^{-1}) \tag{2.5}$$

In order to apply to Eq. (2.5) the convolution theorem for an integral Laplace transformation, the function

$$\Phi(\sigma,t) = L^{-1}(pX_p(a+\sigma p^{-1})) \tag{2.6}$$

is introduced, where L^{-1} is the operator of the inverse Laplace transformation. Then

$$L^{-1}[pX_p(a+\sigma p^{-1})K_0(|\xi-\sigma|)] = \Phi(\sigma,t)K_0(|\xi-\sigma|) \tag{2.7}$$

and, after applying the convolution theorem, we have

$$L^{-1}[pX_p(a+\sigma p^{-1})R_p(|\xi-\sigma|p^{-1})] =$$
$$= L^{-1}[p^2X_p(a+\sigma p^{-1})R_p(|\xi-\sigma|p^{-1})p^{-1}] =$$
$$= \int_0^t \frac{\partial\Phi(\sigma,\tau)}{\partial\tau} Z(|\xi-\sigma|, t-\tau)d\tau \tag{2.8}$$

where

$$\frac{\partial\Phi(\sigma,\tau)}{\partial\tau} = L^{-1}[p^2X_p(a+\sigma p^{-1})]$$
$$Z(|\xi-\sigma|, t-\tau) = L^{-1}[R_p(|\xi-\sigma|p^{-1})p^{-1}] = \tag{2.9}$$
$$= \int_0^A L^{-1}\left[\cos(\lambda|\xi-\sigma|p^{-1})p^{-1}\left(I_2(\sqrt{\lambda^2+p^2})K_2(\sqrt{\lambda^2+p^2}) - \frac{1}{\sqrt{\lambda^2+p^2}}\right)\right]d\lambda$$

In the second relation of (2.9), the convolution theorem is again used taking into account the fact that the originals are found by means of formulae 5.8(2), 5.1(5) and 5.3(35) from [9]:

$$L^{-1}\left(\cos(\lambda|\xi-\sigma|p^{-1})p^{-1}\right) = \mathrm{ber}\left(2\sqrt{\lambda|\xi-\sigma|}\sqrt{t}\right)$$
$$L^{-1}\left(I_2(\sqrt{\lambda^2+p^2})K_2(\sqrt{\lambda^2+p^2})\right) =$$
$$= L^{-1}\left(\int_0^{\pi/2} K_0(2\sqrt{\lambda^2+p^2}\cos x)\cos 2x\right)dx =$$
$$= \int_0^{\pi/2}\left(\frac{1}{\sqrt{t^2-\cos^2 x}} - \lambda\int_0^t \frac{1}{\sqrt{t^2-\cos^2 x-u^2}}J_1(\lambda u)du\right)\cos 2x dx = Q(\lambda, t)$$
$$L^{-1}\left(\frac{1}{\sqrt{\lambda^2+p^2}}\right) = J_0(\lambda t)$$

Finally, instead of the second relation of (2.9), we have

$$Z(|\xi-\sigma|, t) = \int_0^A\int_0^t \mathrm{ber}\left(2\sqrt{\lambda|\xi-\sigma|}\sqrt{t}\right)(Q(\lambda, t-\tau) - J_0(\lambda(t-\tau)))d\lambda d\tau \tag{2.10}$$

Then, in (2.8), integration by parts is carried out, after which, using relations (2.7) and (2.8), Eq. (2.5) in the space of originals can be written in the form

$$\left(1-\frac{d^2}{d\xi^2}\right)\int_0^\infty \Phi(\sigma,t)K_0(|\xi-\sigma|)d\sigma+$$

$$+\left(1-\frac{d^2}{d\xi^2}\right)\int_0^\infty\int_0^t \Phi(\sigma,\tau)L(|\xi-\sigma|,t-\tau)d\tau d\sigma=-\tau^0_\varphi(\xi,t);\quad a\leqslant\zeta<\infty,\quad 0\leqslant t<\infty \tag{2.11}$$

$$L(|\varsigma-\sigma|,t-\tau)=\frac{\partial Z(|\varsigma-\sigma|,t-\tau)}{\partial\tau}$$

An approximate solution of integrodifferential equation (2.11) is constructed by combining the method of orthogonal polynomials [6] and time sampling. The latter means that the time interval $[0, T]$, in which the stress state of the elastic medium is investigated, is divided into intervals $[\tau_k, \tau_{k+1}]$ with a step $h = T/N$; $\tau_k = kT/N$ $(k = 1, 2, \dots, N)$, while the integral with respect to the variable τ is replaced by Simpson's quadrature formula

$$\left(1-\frac{d^2}{d\varsigma^2}\right)\int_0^\infty \Phi_n(\sigma)K_0(|\varsigma-\sigma|)d\sigma+$$

$$+\left(1-\frac{d^2}{d\varsigma^2}\right)\sum_{k=1}^N A_k\int_0^\infty \Phi_k(\sigma)L(|\varsigma-\sigma|,t_n-\tau_k)d\sigma=-\tau^0_{n\varphi}(\varsigma),\quad n=1,2,\dots,N \tag{2.12}$$

where $\Phi_k(\sigma) = \Phi(\sigma, \tau_k)$, and A_k are the coefficients of Simpson's quadrature formula.

To construct the approximate solution of system of equations (2.12), the method of orthogonal polynomials can be used. Here it is necessary to employ the eigenvalue relation [1]

$$\left(1-\frac{d^2}{d\xi^2}\right)\frac{1}{\pi}\int_0^\infty e^{-\sigma}\sqrt{\sigma}L_n^{1/2}(2\sigma)K_0(|\xi-\sigma|)d\sigma=\frac{\sqrt{2}}{n!}\Gamma\left(n+\frac{3}{2}\right)e^{-\xi}L_n^{1/2}(2\xi)$$

$$0\leqslant\xi\leqslant\infty,\quad n=0,1,2,\dots$$

where $L_n^{1/2}(z)$ are Chebyshev–Laguerre polynomials. According to the procedure of this method, the solution of system of equations (2.12) is constructed in the form

$$\Phi_n(\sigma)=\sum_{l=0}^\infty \sqrt{\sigma}e^{-\sigma}L_l^{1/2}(2\sigma)\Phi_l^{(n)} \tag{2.13}$$

after which system (2.12) is reduced to a series $(n = 1, 2, \dots, N)$ of infinite systems of linear algebraic equations

$$Y_l\Phi_l^{(n)}-\sum_{k=1}^N A_k\sum_{m=0}^\infty \Phi_m^{(k)}B_{kml}=F_l^{(n)},\quad l=0,1,2,\dots;\quad n=1,2,\dots,N \tag{2.14}$$

where

$$Y_l=\frac{\Gamma(3/2+l)}{l!},\quad B_{kml}=\int_0^\infty\int_0^\infty \sqrt{\xi}e^{-\xi}L_l^{1/2}(2\xi)\sqrt{\sigma}e^{-\sigma}L_m^{1/2}(2\sigma)L_k(|\xi-\sigma|)d\sigma d\xi$$

$$F_l^{(n)}=-\int_0^\infty \sqrt{\xi}e^{-\xi}L_l^{1/2}(2\xi)\tau^0_{n\varphi}(\xi)d\xi$$

Thus, to determine $\Phi_l^{(1)}$, we obtain the system

$$Y_l\Phi_l^{(1)}-\sum_{m=0}^\infty \Phi_m^{(1)}B_{ml}=F_l^1,\quad l=0,1,2,\dots \tag{2.15}$$

Using the values of $\Phi_l^{(1)}$ found from system (2.15), the subsequent values $\Phi_l^{(2)}$, $\Phi_l^{(3)}$, ... are determined from the recurrence relation

$$Y_l\Phi_l^{(j)} - \sum_{m=0}^{\infty} \Phi_m^{(j)} B_{ml} = F_l^{(j)} - \sum_{k=1}^{j-1} A_k \sum_{m=0}^{\infty} \Phi_m^{(k)} B_{km}, \quad l = 0,1,2,\ldots; \quad j = 2,3,\ldots,N$$

Thus, for the specific value $\Phi_l^{(i)} = \Phi_l (l = 0, 1, \ldots)$ we obtain an infinite system of linear algebraic equations of the form (2.15). To prove the convergence of each such system and the ability to use the reduction method, it is necessary to prove the convergence of the series

$$Q_1 = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} |B_{ml}|^2, \quad Q_2 = \sum_{l=0}^{\infty} \left|f_l^{1/2}\right|^2; \quad f_l^{1/2} = \int_0^{\infty} \tau'(\xi)e^{-\xi}\sqrt{\xi}L_l^{1/2}(\xi)d\xi$$

For the proof, formula 1.14.3(8) from [10] will be necessary:

$$\int_0^x t^{\alpha}e^{-t}L_n^{\alpha}(t)dt = \frac{1}{n}x^{\alpha+1}e^{-x}L_{n-1}^{\alpha+1}(x)$$

We introduce the notation

$$h_n^{1/2} = \int_0^{\infty} h(x)\sqrt{x}e^{-x}L_n^{1/2}(x)dx$$

Integrating by parts in this relationship we obtain

$$h_n^{1/2} = -\frac{1}{n}\int_0^{\infty} \frac{\partial h(x)}{\partial x} x^{3/2}e^{-x}L_{n-1}^{3/2}(x)dx = -h_{n-1}^{3/2}(n)^{-1} \tag{2.16}$$

We will prove the convergence of the series Q_2. Taking formula (2.16) into account, we have

$$f_l^{1/2} = -\frac{1}{l}\int_0^{\infty} f'(\xi)\xi^{3/2}e^{-\xi}L_{l-1}^{3/2}(\xi)d\xi = -\frac{1}{l}f_{l-1}^{3/2}$$

Then Parseval's equality for the series will have the form

$$\sum_{k=0}^{\infty} \left|f_k^{3/2}\right|^2 (N^{3/2})^{-1} = \int_0^{\infty} \sqrt{x}e^{-x}|F'(x)|^2 dx$$

The integral on the right-hand side exists and is finite by virtue of formulae (1.11), which defines the function $F(x)$. This means that the series on the left-hand side also converges.

To prove the convergence of the series Q_1, we represent it in the form of two series

$$Q_1 = Q_1^0 + Q_1^1, \quad Q_1^0 = \sum_{l=0}^{\infty} |B_{0l}|^2, \quad Q_1^1 = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} |B_{jl}|^2$$

where

$$B_{0l} = \int_0^{\infty}\int_0^{\infty} \operatorname{ber}(c\sqrt{|\xi-\sigma|})\sqrt{\xi}e^{-\xi}L_l^{1/2}(\xi)\sqrt{\sigma}e^{-\sigma}d\sigma d\xi = \int_0^{\infty} \sqrt{\sigma}e^{-\sigma}g_l^{1/2}(\sigma)d\sigma$$

$$g_l^{1/2}(\sigma) = \int_0^{\infty} \operatorname{ber}(c\sqrt{|\xi-\sigma|})\sqrt{\xi}e^{-\xi}L_l^{1/2}(\xi)d\xi = -\frac{1}{l}g_{l-1}^{3/2}(\sigma)$$

(relation (2.16) was taken into account). Then, Parseval's equality will have the form

$$\int_0^{\infty}\int_0^{\infty} \left|\frac{\partial \operatorname{ber}(c\sqrt{|\xi-\sigma|})}{\partial\xi}\right|^2 \xi^{3/2}e^{-\xi}L_{l-1}^{3/2}(\xi)d\xi d\sigma = \sum_{l=0}^{\infty} (N_l)^{-1}\int_0^{\infty} \left|g_l^{3/2}(\sigma)\right|^2 d\sigma$$

$$N_l = \Gamma\left(\frac{5}{2}+l\right)[l!(5+l)]^{-1}$$

Using the Cauchy inequality, we obtain the limit

$$|B_{0l}|^2 \leqslant \frac{1}{4l^2}\int_0^\infty \left|g_l^{3/2}(\sigma)\right|^2 d\sigma$$

Thus, the series Q_1^0 converges. To prove the convergence of the series Q_1^1, we use the fact that, according to formula (2.16), $d_{jl} = d_{j-1,l}/(2lj)$, and we assume that $j = k + 1$ in the expression for series Q_1^1.

3. THE CRACK-DISTURBED WAVE FIELD

The Laplace transform of the displacement, according to (1.7) and (1.8) and notation (1.11), has the form

$$u_p^1(r,z) = \int_0^\infty X_p(a+\eta R)\int_0^\infty \cos\lambda(z-\eta)\left[G_{p\lambda}(r,R) - \frac{\partial}{\partial R}G_{p\lambda}(r,R)\right]d\lambda d\eta \tag{3.1}$$

$|z| < \infty,\ 0 \leqslant r < \infty$

Before transferring to the original, (2.4) is substituted into relation (3.1), and then (in order to obtain function (2.6) before using the convolution theorem), the integrand is multiplied and divided by p. As a result, the required wave field takes the form

$$u^1(r,z,t) = \int_0^\infty\int_0^t \Phi(\sigma,t)L^{-1}\left(\cos\lambda(z-\sigma p^{-1}R)p^{-2}\left(G_{p\lambda}(r,R) - \frac{\partial}{\partial R}G_{p\lambda}(r,R)\right)\right)d\lambda d\tau \tag{3.2}$$

To calculate the originals in (3.2), tabulated values 5.8(8) from [3] are used:

$$\begin{aligned} &L^{-1}(\cos\lambda(z-\sigma p^{-1})p^{-2}) = \cos\left(\lambda z - \frac{3\pi}{4}\right)\left(\tfrac{1}{2}\sqrt{\lambda\sigma}\right)^{-1}\sqrt{t}\,\mathrm{ber}_1(\sqrt{\lambda\sigma}\sqrt{t}) + \\ &+\sin\left(\lambda z - \frac{3\pi}{4}\right)\left(\tfrac{1}{2}\sqrt{\lambda\sigma}\right)^{-1}\sqrt{t}\,\mathrm{ber}_1(\sqrt{\lambda\sigma}\sqrt{t}) \equiv X_1(z,\lambda,t) \\ &L^{-1}(G_{p\lambda}(r,R)) = \\ &= \int_0^\infty \alpha J_1(\alpha R)J_1(\alpha r)2\pi(\sqrt{\lambda^2+\alpha^2})^{-1/2}\sqrt{t}J_{1/2}(\sqrt{\lambda^2+\alpha^2}t)d\alpha \equiv X_2(r,R,\lambda,t) \end{aligned} \tag{3.3}$$

Taking into account that $J_{1/2}(x) = (2/(\tau x))^{1/2}\sin x$, and using the integral representation for the Bessel functions ([8], formula 7.14.2(61))

$$2\pi J_1(\alpha R)J_1(\alpha r) = -\int_{-\pi}^{\pi} e^{i\theta}\frac{r-\mathrm{R}e^{-i\theta}}{r-\mathrm{R}e^{i\theta}}J_2(\alpha\omega)d\theta,\quad \omega = \sqrt{r^2+R^2-2rR\cos\theta}$$

we obtain

$$X_2(r,R,\lambda,t) = L^{-1}\left(G_{p\lambda}(r,R)\right) = \frac{2}{\sqrt{\pi}}\int_{-\pi}^{\pi} e^{i\theta}\frac{r-\mathrm{R}e^{-i\theta}}{r-\mathrm{R}e^{i\theta}}\frac{1}{\sqrt{t^2-\omega^2}}\cos(\lambda\sqrt{t^2-\omega^2})d\theta$$

Here, we have used the value of the tabulated integral ([8], formula 2.12.23(8)).

Finally, a formula is obtained for carrying out calculations

$$u^1(r,z,t) = \int_0^\infty\int_0^t \Phi(\sigma,\tau)[M(r,R,z,\lambda,t-\tau) - \frac{\partial}{\partial R}M(r,R,z,\lambda,t-\tau)]d\tau d\lambda$$

$$M(r,R,z,\lambda,t) = \int_0^t X_1(z,\lambda,t-\tau) * X_2(r,R,\lambda,\tau)d\tau$$

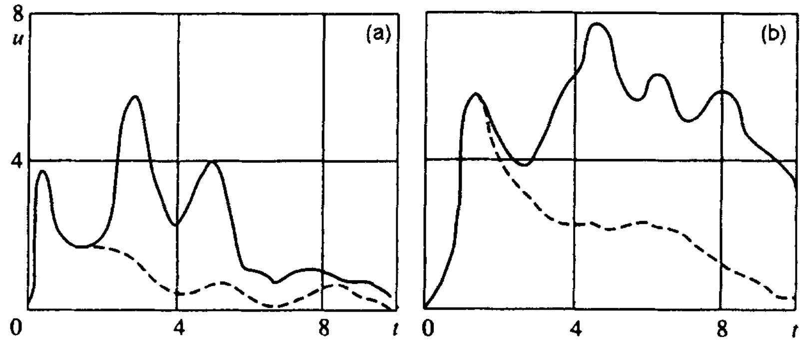

Fig. 1

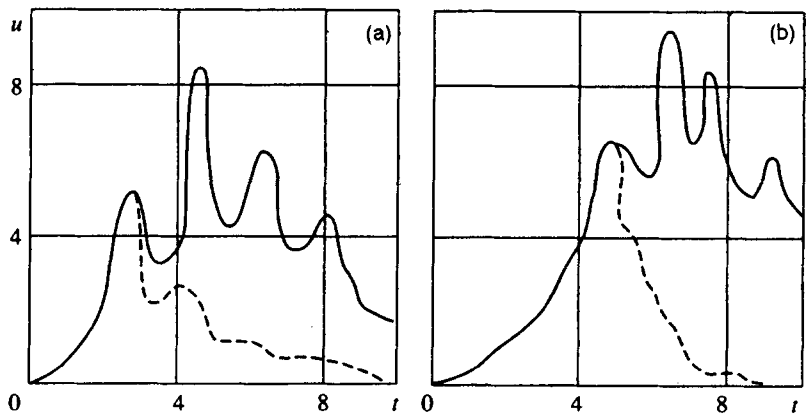

Fig. 2

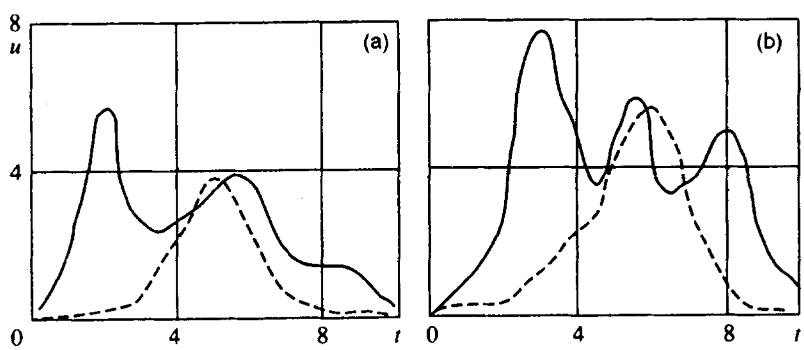

Fig. 3

The time-dependence of the displacements at fixed points for different distances a from the centre of rotation to the plane passing through the edge of the crack was calculated.

For example, for $a = R$, the results obtained are shown in Figs 1–3, which illustrate the nature of the change with time of the displacements when there is a crack (the continuous curves) or when there is no crack (the dashed lines) in the medium. Graphs of the displacement of the point with the coordinate $z = R/2$ (Fig. 1), $z = R$ (Fig. 2) and $z = 3R/2$ (Fig. 3) as a function of time t are shown. The left-hand parts of the figures (a) correspond to points at which $r = 0$, and the right-hand parts (b) to points at which $r = 2R$. It can be seen that, for points no further than the edge of the crack from the centre of rotation, the first peak of the displacement is observed at the same time, irrespective of whether a crack is present or not, but when there is a crack, the maximum values of the displacements are reached later. The coincidence of the values of the displacements at a certain initial instant of time (irrespective of whether a crack is present or not) is due to the influence of the crack still not being felt because of the fact that the disturbances have not yet reached it. The subsequent appearance of peaks in the values of the displacements in Figs 1 and 2 is due to the fact that the crack begins partially to screen the disturbances created by the centre of rotation. This may explain why, when the point of observation is located further away than the edge of the crack (Fig. 3), the peak in the values of the displacements when a crack is present occurs earlier than when there is no crack.

For these points, graphs of the change in the stresses as a function of time were also plotted, the structure of which is similar, and are therefore not given here. To investigate the dependence of the wave fields on the distance of the centre of rotation to the edge of the crack, similar graphs were plotted for $a = 2R$. Since the above properties of the wave field were retained here (only the peaks in the values of the displacements are observed later in time and their values are lower), these graphs again are not given here.

4. DERIVATION OF A FORMULA FOR THE STRESS INTENSITY FACTOR

In the accepted notation of fracture mechanics, we have a formula for the stress intensity factor (SIF) which, in Laplace transforms, can be written as follows:

$$K_{\rm III}^p = \lim_{z\to a-0} \tau_{\varphi p}(R,z)\sqrt{2\pi(a-z)}$$

Taking into account the substitution $z = a + xR$ and substitution (2.4), this formula becomes

$$K_{\rm III}^p = \sqrt{2\pi}\lim_{\xi\to -0} \tau_{\varphi}(R,a+\xi p^{-1})\sqrt{|\xi|p^{-1}} \tag{4.1}$$

By inverting (4.1), we obtain an expression for the SIF which, taking into account the time sampling above, we will write in the form

$$K_{\rm III}^n = K_{\rm III}(t_n) = \sqrt{2}\lim_{\xi\to -0}\sum_{j=1}^{N} B_j\tau_\varphi(\xi,\tau_j)(t_n-\tau_j)^{1/2}\sqrt{|\xi|}, \quad n=1,2,\dots,N \tag{4.2}$$

where

$$\tau_\varphi^j(\xi) = \left(1-\frac{d^2}{d\varsigma^2}\right)\int_0^\infty \Phi_j(\sigma)K_0(|\varsigma-\sigma|)d\sigma +$$

$$+\left(1-\frac{d^2}{d\varsigma^2}\right)\sum_{k=1}^{N} A_k\int_0^\infty \Phi_k(\sigma)L(|\varsigma-\sigma|,t_j-\tau_k)d\sigma + \tau_{j\varphi}^0(\varsigma), \quad j=1,2,\dots N \tag{4.3}$$

We will calculate the limit in (4.2). The last two terms in (4.3) will make no contribution to the SIF by virtue of formulae (1.11) and (2.11). In the first term in (4.3), series (2.13) is substituted under the integral sign. We obtain

$$K_{\rm III}^n = \sqrt{2}\lim_{\xi\to -0}\sum_{j=1}^{N} B_j\sqrt{|\xi|}\left(1-\frac{d^2}{d\xi^2}\right)\int_0^\infty\sum_{k=0}^{\infty}\sqrt{\sigma}e^{-\sigma}\times$$

$$\times L_j^{1/2}(2\sigma)\Phi_k^{(j)}K_0(|\xi-\sigma|)d\sigma(t_n-\tau_j)^{1/2} \tag{4.4}$$

It was shown in [1] that

$$J_j^{(1)}(\xi) = \left(1-\frac{d^2}{d\xi^2}\right)\frac{1}{\pi}\int_0^\infty \sqrt{\sigma}e^{-\sigma}L_j^{1/2}(2\sigma)K_0(|\xi-\sigma|)d\sigma =$$

$$= \frac{\Gamma(1/2+1+j)}{j!\pi}I_1^{-1}(\xi), \quad \xi<0, \quad I_1^{-1}(\xi) = \int_0^\infty \frac{t^{j+1/2}e^{t\xi}}{(t+2)^{j+1}}dt$$

$$\lim_{\xi\to -0}\sqrt{|\xi|}\,J_j^{(1)}(\xi) = \frac{\Gamma(j+3/2)}{j!\pi}\lim_{\xi\to -0}\sqrt{|\xi|}\,I_n^{-1}(\xi) = \frac{\Gamma(j+3/2)}{j!(-1)^{j+2}\sqrt{2\pi}}.$$

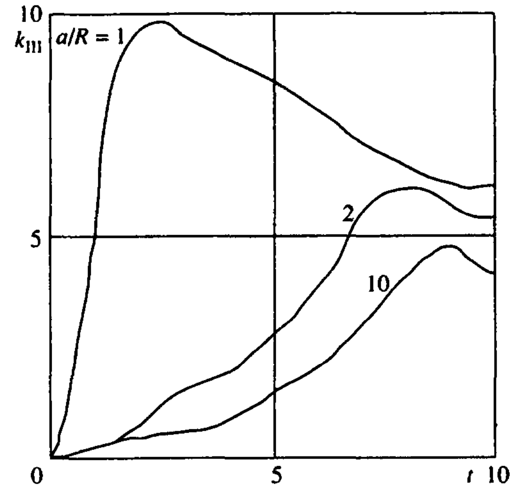

Fig. 4

Finally, after substituting this result in (4.4), we obtain

$$K_{III}(t_n) = \sqrt{\frac{2}{\pi}} \sum_{j=1}^{N} B_j \frac{\Gamma(j+3/2)}{j!} (t_n - \tau_j)^{1/2} \sum_{k=0}^{\infty} \Phi_k^{(j)}$$

The coefficients $\Phi_k^{(j)}$ are determined from system (2.14).

Figure 4 shows graphs of the time-dependence of the SIF for different values of a/R.

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